# Math 222A Lecture 21 Notes

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## 1 Fourier Transforms of Periodic Functions and Local Solvability of Partial Differential Operators

### 1.1 Fourier transforms of periodic functions

A function $f$ is periodic if

$$f(x) = f(x+a)$$

for some $a$ and for all $x$.

**Definition 1.1.** $f \in \mathcal{D}'$ is **periodic** of period $a$ if

$$f(\phi) = f(\phi(\cdot + a))$$

Suppose $f$ is periodic; what can we say about $\widehat{f}$? Recall that for functions,

$$\widehat{f}(\cdot + a) = e^{ia\cdot\xi}\widehat{f}.$$

Using the periodic condition, write this as the multiplication

$$\widehat{f}(1 - e^{ia\xi}) = 0.$$

Note that $1 - e^{ia\xi} \neq 0$ unless $\xi = \frac{2\pi n}{a}$. Then $\operatorname{supp} \widehat{f} \subseteq \frac{2\pi n}{a}\mathbb{Z}$. As an analogy look at the condition $xf = 0 \implies f = c\delta_0$; here, we have zeros at many points. So we conclude that

$$\widehat{f} = \sum_n c_n \delta_{\frac{2\pi n}{a}}.$$

**Theorem 1.1.** *The coefficients $c_n$ are the Fourier coefficients for $f$ in the interval $[0, a]$, and*

$$f(x) = \sum_n c_n e^{\frac{2\pi i}{a} n}.$$

Here, we have ignored the factors of $2\pi$.

**Remark 1.1.** We can multiply $f$ by $e^{-\frac{2\pi i}{a}m}$ and integrate from 0 to $a$ to get

$$c_n = \int f(x)e^{-\frac{2\pi i}{a}mx}.$$

**Example 1.1.** The simplest periodic distribution is

$$f_a = \sum_n \delta_{na}.$$

Then

$$\widehat{f_a} = \sum_n c_n \delta_{\frac{2\pi}{a}n}.$$

If we write

$$f_a(1 - e^{\frac{2\pi i x}{a}}) = 0,$$

then we get

$$\widehat{f_a} = \widehat{f_a}(\cdot + \tfrac{2\pi}{a}).$$

Thus, all the $c_n$s are the same. So

$$\widehat{f_a} = c_a \sum_n \delta_{\frac{2\pi n}{a}} = c_a f_{\frac{2\pi}{a}}.$$

What is $c_a$? Apply this to a Schwarz function: $\widehat{f}(\phi) = f(\widehat{\phi})$ by definition, so

$$c_a \sum_{n \in \mathbb{Z}} \phi(\tfrac{2\pi n}{a}) = \sum_{m \in \mathbb{Z}} \widehat{\phi}(ma).$$

This is called the **Poisson summation formula**.

Now what happens if we replace $\phi$ by $\phi e^{ix\cdot\xi_0}$? Then $\widehat{(}\xi)$ becomes $\widehat{\phi}(\xi - \xi_0)$. The Poisson summation formula gives

$$c_a \sum_n \phi(\tfrac{2\pi n}{a}) e^{i\frac{2\pi n}{a}\xi_0} = \sum \widehat{\phi}(ma - \xi_0).$$

The dependence of $\xi_0$ on the left hand side is simple. Integrate to get

$$\begin{aligned}
\underbrace{\int_0^a \sum_n \phi(\tfrac{2\pi n}{a}) e^{i\frac{2\pi n}{a}\xi_0}\, d\xi_0}_{=ac_a\phi(0)} &= \int_0^a \sum_m \widehat{\phi}(ma - \xi_0)\, d\xi_0 \\
&= \int \widehat{\phi}(\xi)\, d\xi \\
&= \widehat{\phi}(1) \\
&= \phi(\tfrac{1}{\sqrt{2\pi}}\delta_0)
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}}\phi(0).$$

Accounting for the constants we ignored before, we get

$$c_a = \frac{1}{2\pi a}.$$

**Remark 1.2.** We can use the Poisson summation formula to compute all sorts of series. Recall that $\mathcal{F}(\frac{1}{1+x}) = ce^{-|\xi|}$ (perhaps omitting constants). Choose $a = 2\pi$. The Poisson summation formula tells us that

$$\sum_{m\in\mathbb{Z}} \frac{1}{n^2+1} = \sum_m e^{-2\pi|m|} = \frac{2}{1-e^{2\pi}} - 1,$$

where we have ignored the constants.

## 1.2 Local solvability of partial differential operators

Let $P(D)$ be our partial differential operator with constant coefficients.

**Definition 1.2.** $P(D)$ is **solvable** if for each $f$, the equation $P(D)u = f$ admits at least one solution.

If $f \in \mathcal{D}'$, then $u \in \mathcal{D}'$. If $f \in \mathcal{S}$, then $u \in \mathcal{S}$. In general, the regularity of $f$ and $u$ will be related, so when we say $P(D)$ is solvable, we specify a class of functions $f$.

**Definition 1.3.** $P(D)$ is **locally solvable** if for each $f \in \mathcal{E}'$, there exists a solution $u \in \mathcal{D}'$ in a neighborhood of the support of $f$.

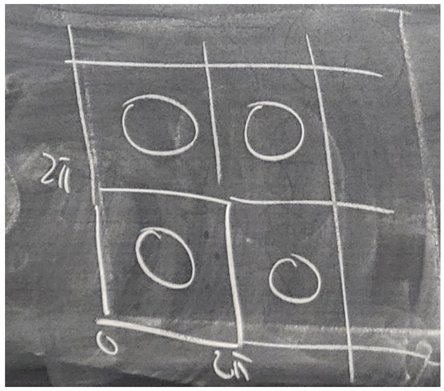
If $u \in \mathcal{E}'$, then $P(\xi)\widehat{u}(\xi) = \widehat{f}(\xi)$ for $\xi \in \mathbb{C}^n$. Here is a narrower version, which we may regard as the "real definition" of local solvability:

**Definition 1.4.** $P(D)$ is **locally solvable** if for each $x_0$, there is an $\varepsilon > 0$ such that if $\operatorname{supp} f \subseteq B(x_0, \varepsilon)$, then a solution exists.

For today, we will deal with the first, more relaxed definition.

**Theorem 1.2.** *Every constant coefficient partial differential operator is locally solvable (in the relaxed sense).*

*Proof.* Suppose $f$ is supported in $B \subseteq [0, 2\pi]^n$. Take $\widetilde{f}$ to be the periodic extension of $f$, and look for a periodic solution $\widetilde{u}$ to $P(D)\widehat{u} = \widetilde{f}$.




What does this periodization do? Originally, $P(D)u = f$ gives $P(\xi)\widehat{u} = \widehat{f}$, so $\widehat{u} = \frac{1}{P(\xi)}\widehat{f}$. However, this has issues because $P(\xi)$ can have issues. In the periodic case, we know

$$\widehat{\widetilde{f}}(\xi) = \sum_{m \in \mathbb{Z}^n} f_m \delta_m,$$

$$\widehat{\widetilde{u}}(\xi) = \sum_{m \in \mathbb{Z}} u_m \delta_m.$$

We need $P(m)u_m = f_m$, which gives

$$u_m = \frac{f_m}{P(m)}, \qquad m \in \mathbb{Z}^n.$$

The advantage is that we only $P(m) \neq 0$ on lattice points $m \in \mathbb{Z}^n$. However, the Fourier transform is defined for temperate distributions, so we need about on $\frac{f_m}{P_m}$. More precisely, we need a bound

$$|P(m)| \geq (1 + |m|)^{-N}$$

What if $P$ has zeroes on the lattice points? Make the change of notation $f \mapsto fe^{ix\cdot\xi} = g$, so $u \mapsto ue^{ix\xi} = v$. We can ask this question for the phase-shifted variables. To study our equation, we need to expand

$$P(D)u = P(D)(ve^{-ix\cdot\xi}).$$

To use the Leibniz rule, note that,

$$\begin{aligned} D_j(ve^{-ix\xi}) &= Dve^{-ix\cdot\xi} + vD_je^{-ix\cdot\xi} \\ &= e^{-ix\cdot\xi}(D_jv - v\xi_j) \end{aligned}$$

$$= e^{-ix\cdot\xi}(D_j - \xi_j)v,$$

We can write this as $e^{ix\cdot\xi}D_j e^{-ix\xi} = D_j - \xi_j$, which we may think of as a **conjugation**. Referring to our equation, we get

$$\begin{aligned} P(D)u &= P(D)(ve^{ix\ cdot\xi} \\ &= e^{-ix\cdot\xi}p(D-\xi)v \\ &= f, \end{aligned}$$

which tells us that we have replaced $P(D)u = f$ with

$$P(D-\xi)v = g.$$

So we only need to solve the new periodic problem is to define

$$v_m = \frac{g_m}{P(m-\xi)}, \qquad m \in \mathbb{Z}.$$

Now we only need to find some $\xi \in [0,1]^n$ such that

$$|P(m-\xi)| \geq (1+|m|)^{-N} \qquad \forall m.$$

The following lemma tells us we can find such a $\xi$.

**Lemma 1.1.** *If $\delta$ is small enough, then*

$$\int \frac{1}{(P(\eta))^\delta} \frac{1}{(1+|\eta|)^N}\, d\eta < \infty.$$

*Proof.* In 1 dimension, use partial fractions. Then reduce any number of dimensions to the 1-dimensional case. □

How does this help us? Write $\eta = m + \xi$ with $m \in \mathbb{Z}^n$ and $\xi \in [0,1]^n$. Then

$$\int_\xi \sum_m \frac{1}{P(m-\xi)|^\delta} \frac{1}{(1+|m|)^N}\, d\eta < \infty.$$

So for almost every $\xi$,

$$\sum_m \frac{1}{|P(m-\xi)|^\delta} \frac{1}{(1+|m|)^N} = M < \infty.$$

This tells us that

$$|P(m-\xi)| \geq M^{-1/d}(1+|m|)^{-N/\delta},$$

which is exactly the relation we want to have. □